

Transience of a Server with Greedy Strategy on the Real Line

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Abstract

We consider a single-server system with stations at each point of the real line. The customer arrivals are given by a Poisson point processes on the space-time half plane. The server adopts a greedy routing mechanism, traveling towards the nearest customer, and ignoring new arrivals while in transit. We study the trajectories of the server and show that its asymptotic position diverges logarithmically in time.

1 Introduction

In this paper we study the asymptotic behavior of a greedy single-server system on the real line, where customers arrive as a Poisson rain. The system is described as follows. Initially there is a Poisson field of customers in \mathbb{R} and the server starts at $x = 0$. Customers arrive as a Poisson point process in the space-time $\mathbb{R} \times \mathbb{R}_+$ with intensity 1. When not serving, the server chooses the nearest customer and travels towards it at speed $0 < v \leq \infty$, ignoring new arrivals. The service then takes T units of time with $\mathbb{E}T = 1$, after which the customer leaves the system. This is a common example of a routing mechanism that depends on the system state, and targeting the nearest customer is known as a *greedy strategy*.

The understanding of customer-server systems with a large number of service stations having some spacial structure is often obscured by their combinatorial aspects. On the other hand, continuous-space approximations are more transparently

described. However, systems with greedy routing strategies in the continuum are extremely sensitive to microscopic perturbations, and their rigorous study represents a challenging problem (see [4] and references therein). The question of stability, which is fundamental in many cases, is inadequate for the greedy server on the infinite real line. The latter serves rather as a model that captures the local behavior of other systems where the total arrival rate is finite, such as the greedy server on the circle introduced in [1].

Let \mathcal{S}_t denote the server position at time t .

Theorem 1. *Assume that $\mathbb{E}e^{\alpha T} < \infty$ for some $\alpha > 0$. Then for any $v > 0$ the greedy server on the real line is transient. Moreover,*

$$\frac{\mathcal{S}_t}{\log t} \rightarrow \pm 1$$

with probability 1/2 each.

Remark 1. We present a proof for $T = 1$ and $v = \infty$ in this paper to simplify the presentation. The same proof works for any $v > 0$ and i.i.d. service times T with an exponential moment after a few straightforward adaptations, which we omit as they would require more cluttered notation without introducing any new idea. On the other hand, it is important that the arrivals form Poisson Process in space-time, and that they are independent of the service times.

Remark 2. Assume that at time 0– the set of waiting customers is distributed as a Poisson Point Process with intensity $\mu(x)dx$, for some non-negative bounded measurable function μ with $\int \mu = \infty$, and with an additional deterministic finite set of points. Then Theorem 1 remains true (with essentially the same proof), except for the lack of symmetry in the probabilities of \mathcal{S}_t diverging to $+\infty$ or $-\infty$.

Remark 3. There is a dynamic version of the greedy server, where new arrivals are not ignored while the server is traveling. This variation might be studied by similar arguments, but the dynamic mechanism introduces some extra complications that will not be considered here.

Heuristically, the asymptotics described by Theorem 1 is what one should expect to happen, assuming that the server will indeed move most of the times in the same direction. Suppose that all of the first N customers were found to the right of the server. The typical distance between the server and the next customer to the right is about $\frac{1}{N}$, because customers have been arriving to this region for about N time units. To the left of the server there are regions of size about $\frac{1}{N-1}$, $\frac{1}{N-2}$, $\frac{1}{N-3}$, etc., where

the arrival of customers is rather recent: they must have happened during the last 1, 2, 3, etc., units of time. If the server is eventually moving only to the right (or the excursions to the left are very sparse in time), the server position S_N should therefore diverge as $\log N$. However, the probability that the next customer is found to the left of the server is about $\frac{C}{N}$, which implies that it will happen some time in the future. In fact, the server will make an excursion of length $\frac{c}{N}$ to the left for infinitely many N , for any constant c . Nevertheless, the probability that the two next customers are both to the left is about $\frac{C}{N^2}$. One may thus push this argument and show that indeed, with positive probability, the system will never produce microscopic scenarios which could – together with this local routing mechanism – cause important changes in the server's course.

The same asymptotics has been obtained in [2] for a continuous model in \mathbb{R} where there is no greedy mechanism, and the server is always moving to the right. A model with a discrete set of stations \mathbb{Z} and greedy routing scheme was considered in [3], where the possibility of backward jumps is eventually overwhelmed by the typical macroscopic configurations within each of the service stations. In the continuous system we show that, unlike the discrete variant, the server does change direction infinitely often, but eventually all the changes in direction are reverted immediately.

Our approach is based on the customers environment as viewed from the server. Namely, it learns only the information that is necessary and sufficient to determine the next movement, and the positions of further waiting customers remain unknown.

This paper is divided as follows. In Section 2 we present the evolution of environment as viewed from the server and study its properties. We state Proposition 1 about the behavior of the server on the real line at specific times (a block argument) and show how it implies Theorem 1. In Section 3 we prove Proposition 1.

Notation We write $a \sim_\epsilon b$ if $\limsup \left| \frac{a}{b} - 1 \right| \leq \epsilon$ and $a \sim b$ if $\frac{a}{b} \rightarrow 1$. Each time C or c (resp. C_ϵ) appears, it denotes a different constant (resp. function of ϵ) that is positive, finite, and universal. $a \vee b$ means $\max\{a, b\}$.

2 The process viewed from the server

We consider a particular construction of the initial state by assuming that there are arrivals during $t \in [-1, 0]$, before the service starts at $t = 0$. This is of course

equivalent to simply starting at $t = 0$ with a Poisson field of points. Let ν denote the random set of arrivals in $\{(x, t) : x \in \mathbb{R}, t > -1\}$.

We want to construct the process by following a progressive exploration of the space-time until finding the mark $(x^*, t^*) \in \nu$ corresponding to the nearest waiting customer, getting as little information as possible about ν . The server is thus unaware of existing customers further than the nearest one, and keeps record of the last time when each point in space was explored in the seek of waiting customers.

Recall that for simplicity we are considering the case $T = 1$ and $v = \infty$, thus the server's position \mathcal{S}_t remains constant on intervals $t \in [n - 1, n)$.

Starting at $t = 0$, each region on the space has potentially witnessed the arrival of customers during 1 unit of time. The first customer is then found at a exponentially-distributed distance, to the left or to the right with equal probabilities. Discovering its position reveals the presence of a point in ν , as well as a region where ν has no points. For the second customer, there is a region in space that has potentially witnessed the arrival of customers during 1 unit of time (namely, the region explored on the previous step), and the complementary region has not been queried during the last 2 units of time. The position of the third customer is already more involved, and the positions of both of the previous customers are important in determining the regions where ν is still unexplored. Yet there is a general description which is amenable to study, which motivates the construction described hereafter.

A *potential* is a piecewise continuous function $u : \mathbb{R} \rightarrow \mathbb{R}$ such that there is a unique point $x_* = S(u)$ where it attains its maximum $\mathcal{M} = \mathcal{M}(u) = u(x_*)$.

Given a pair of positive numbers $w = (E, U)$, where $0 < E < \infty$ and $0 < U < 1$, we define the operator \mathcal{H}_w as follows. Let u be given and take $z > 0$ as the unique number such that

$$\int_{x_* - z}^{x_* + z} (\mathcal{M} - u) dx = E.$$

Let

$$a = \mathcal{M} - u(x_* - z), \quad b = \mathcal{M} - u(x_* + z),$$

choose

$$x^* = \begin{cases} x_* - z, & \text{if } U \in (0, \frac{a}{a+b}], \\ x_* + z, & \text{if } U \in (\frac{a}{a+b}, 1), \end{cases}$$

and finally

$$(\mathcal{H}_w(u))(x) = \begin{cases} \mathcal{M} + 1, & x = x^*, \\ \mathcal{M}, & x \in [x_* - z, x_* + z], \ x \neq x^*, \\ u(x), & \text{otherwise.} \end{cases} \quad (1)$$

Notice that $\mathcal{M}(\mathcal{H}_w(u)) = \mathcal{M}(u) + 1$, $S(\mathcal{H}_w(u)) = x^*$, and $\int_{\mathbb{R}} [\mathcal{H}_w(u) - u] dx = E$.

Start with $u_0^0 : \mathbb{R} \rightarrow \mathbb{R}$ given by $u_0^0(x) = \delta_0(x) - 1$, let $(E_n)_n$ and $(U_n)_n$ be independent i.i.d. sequences of exponential and uniform random variables, and write $w_n = (E_n, U_n)$. Let $u_n^0 = \mathcal{H}_{w_n}(u_{n-1}^0)$ and write $S_n = S(u_n^0)$.

Lemma 1. *The sequence $(S_n)_{n=1,2,\dots}$ defined above has the same distribution as the sequence $(S_{t-})_{t=1,2,\dots}$ given by the positions of the greedy server at integer times.*

The lemma follows from the properties of the Poisson point process ν on

$$\Gamma_u := \{(x, t) : x \in \mathbb{R}, u(x) \leq t < \infty\}.$$

Indeed, consider a progressive exploration at the left and right vertical boundaries of the continuously-expanding region $\{(x, t) : x_* - z \leq x \leq x_* + z, u(x) \leq t \leq \mathcal{M}(u)\}$ as z increases, starting from 0 until finding the first point (x^*, t^*) of ν . The variable E is given by the area of the explored region. The variable U is related to the position of (x^*, t^*) on the union of the two disjoint vertical intervals where this region is growing, and is given by $t^* = \mathcal{M}(u) - |(a+b)U - a|$, $x^* = x_* + z \cdot \text{sgn}[(a+b)U - a]$. By the properties of a Poisson point process, E and U are independent of each other, distributed as standard exponential and uniform variables, regardless of how ν had been explored outside Γ_u .

We now consider some properties of the operators \mathcal{H} . Let $\theta_z u = u(z + \cdot)$. For any potential u , $\mathcal{M}(\theta_z u + c) = \mathcal{M}(u) + c$ and $S(\theta_z u + c) = S(u) - z$. It follows from the definition of \mathcal{H} that

$$\mathcal{H}_w(\theta_z u + c) = \theta_z \mathcal{H}_w(u) + c. \quad (2)$$

A potential u is said to be *centered* if $S(u) = \mathcal{M}(u) = 0$. Define the operator $\Theta^u(\cdot) = \theta_{S(u)}(\cdot) - \mathcal{M}(u)$, so that $\Theta^u(u)$ is centered. For given potentials u and \tilde{u} ,

$$\Theta^{\Theta^u(\tilde{u})} \circ \Theta^u = \Theta^{\tilde{u}}. \quad (3)$$

The natural *shifts* in this evolving sequence of potentials $(u_n^0)_{n \geq 0}$ is given for each k by $(u_n^k)_{n \geq 0}$ defined as $u_n^k := \Theta^{u_1^{k-1}}(u_{n+1}^{k-1})$. Expanding this recursion and using (3)

yields

$$u_n^k = \Theta^{u_1^{k-1}}(\Theta^{u_1^{k-2}}(u_{n+2}^{k-2})) = \Theta^{u_2^{k-2}}(u_{n+2}^{k-2}) = \dots = \Theta^{u_k^0}(u_{n+k}^0).$$

In particular, $u_0^k = \Theta^{u_k^0}(u_k^0)$. Writing

$$\mathcal{H}_n^k = \mathcal{H}_{w_{k+n}} \circ \dots \circ \mathcal{H}_{w_{k+2}} \circ \mathcal{H}_{w_{k+1}},$$

it follows from (2) that $u_n^k = \mathcal{H}_n^k(u_0^k)$. Therefore u_0^k is determined by w_1, w_2, \dots, w_k , whereas $(u_n^k)_{n \geq 0}$ is determined by w_{k+1}, w_{k+2}, \dots and u_0^k itself.

The above properties imply that the evolution of $(u_n^0)_n$ is a homogeneous, translation-invariant, height-invariant, strong Markov chain in the space of potentials. At any moment k , we can take u_k^0 and move the axes so that the origin is placed on its maximum (that is, apply $\Theta^{u_k^0}$), obtaining u_0^k , and from this point on the evolution of $(u_n^k)_n$ is independent of (u_1^0, \dots, u_k^0) , and obeys the same transition rules. Moreover, $(u_n^k)_n$ is related to $(u_{k+n}^0)_n$ by $u_n^k = \Theta^{u_k^0}(u_{k+n}^0)$. An example depicting this construction is shown in Figure 1.

This motivates us to define the evolution of the greedy server model starting from any centered potential u as the initial u_0^0 , not necessarily given by $\delta_0 - 1$. Namely, the system starts at $t = 0$, with customer arrivals in space-time given by a Poisson point process ν on Γ_u . We denote its law by \mathbb{P}^u .

In the proof of Theorem 1 we only use two properties of $u_0^0(x) = \delta_0(x) - 1$. We say that a potential u is *unimodal* if u is non-decreasing on $(-\infty, S(u))$ and non-increasing on $(S(u), +\infty)$. We say that a potential u is *bounded* if $m(u) := \mathcal{M}(u) - \inf_{x \in \mathbb{R}} u(x)$ is finite. Each of these conditions is preserved by the operators of the form \mathcal{H}_ω . Since they are also preserved by θ_z and $u \mapsto u + c$, starting from $u_0^0(x) = \delta_0(x) - 1$ the potentials u_n^k are unimodal and bounded for any k and n .

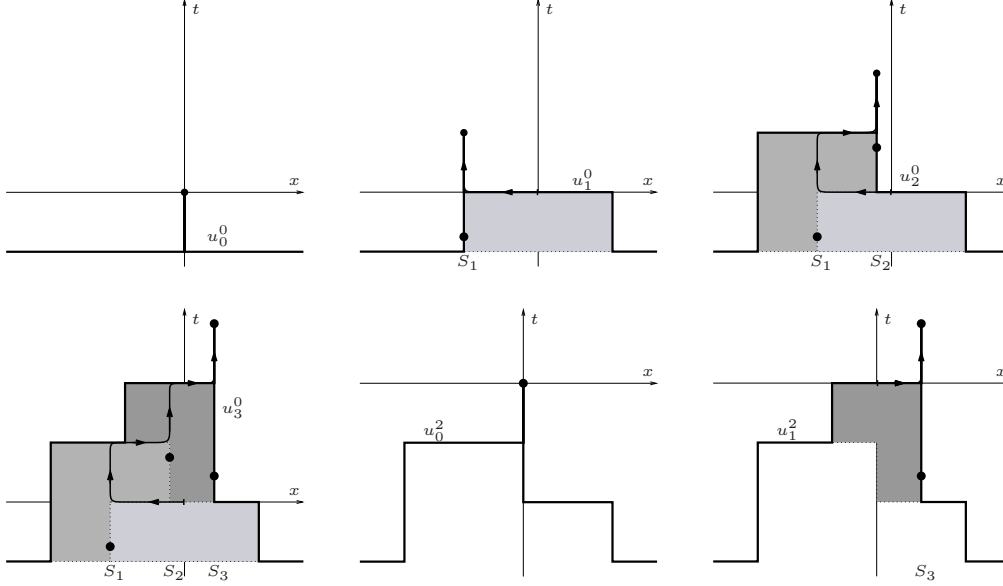


Figure 1: Revealing three points of $\nu \subseteq \mathbb{R} \times (-1, \infty)$ to determine the greedy server's first steps. Before starting, the configuration is unknown on the whole $\nu \subseteq \mathbb{R} \times (-1, \infty)$, represented by the graph $u_0^0(x) = \delta_0(x) - 1$. The nearest customer found at time 0 corresponds to the bold point (x^*, t^*) in the second plot (middle above), where the graph of u_1^0 covers the region that had to be explored in order to find (x^*, t^*) . After serving this customer, the point in ν corresponding to the nearest customer corresponds to a new bold point appearing in the third plot, where the graph of u_2^0 covers the total region explored in these two steps. The server's trajectory is depicted by the arrowed, curly path, and consists of unit service times alternated with instantaneous space displacements. The fourth plot (below, left) shows the three points of ν determining the construction of u_3^0 , the region of $\Gamma_{u_0^0}$ explored, and the path performed by the server during the interval $[0, 3)$. The fifth and sixth plots (below, center and right) depict the Markovian nature of this procedure. At the second customer's departure time, we place the axes on the maximum of u_2^0 , obtaining u_1^2 . Notice that in this picture there is no record of the past trajectory and the location of the other two points also called (x^*, t^*) . It turns out that the potential is enough in order to determine the future evolution, and we find the same point (x^*, t^*) corresponding to the next customer.

Grouping customers Theorem 1 is proved by grouping customers as in the following proposition. Write

$$\ell_j = \lceil 12j^{1/4} + 1 \rceil, \quad j = 0, 1, 2, \dots$$

Proposition 1. *Let u be a centered potential, and suppose that u is unimodal. For any $\epsilon > 0$, there exists $\delta > 0$ depending on ϵ but not on the potential u , and a sequence of stopping times $L_0, L_1, L_2, L_3, \dots$ with the following properties.*

Taking $\sigma = \operatorname{sgn} S_1$, $Z_j = \sigma S_{L_j}$, $N_j = L_j - u(S_{L_j})$, $Q_j = L_{j+1} - L_j$ and $X_j = Z_{j+1} - Z_j$, with probability at least δ we have, for all $j = 1, 2, 3, \dots$,

$$\begin{cases} Q_j^- \leq Q_j \leq Q_j^+, \\ X_j^- \leq X_j \leq X_j^+, \\ Z_{j-1} < \sigma S_n < Z_{j+1}, \quad \text{for } L_j \leq n < L_{j+1}, \end{cases} \quad (4)$$

where

$$\begin{aligned} Q_j^- &= \ell_j, & Q_j^+ &= \ell_j + 1, \\ X_j^- &= (1 - \epsilon) \frac{\ell_j - 1}{N_{j+1}}, & X_j^+ &= (1 + \epsilon) \frac{\ell_j}{N_j}. \end{aligned}$$

In words, Z_j is the server position after serving L_j customers, N_j is the discontinuity in the potential at this moment, and finally X_j measures the displacement in space after serving the next Q_j customers.

The above proposition is proved in the next section. Let us show how it implies the main result.

Proof of Theorem 1. Let ϵ be any positive number. The system starts at time $n_0 = 0$ from the potential u_0^0 , and by Proposition 1, with probability at least δ the events (4) hold for all j , for some sequence of stopping times L_j . If it does not hold for all j , let j_* be the first j for which condition (4) is violated, and call $n_1 = L_{j_*+1}$. Whether (4) occurs or not is determined by $(u_n^0)_{n=0,1,\dots,L_{j_*+1}}$. Since L_{j_*+1} is a stopping time, at time n_1 the system restarts from some unimodal bounded potential $u_0^{n_1}$, ignoring the past history, i.e., conditioned on n_1 and $u_{n_1}^0$, $(u_n^{n_1})_{n \geq 0}$ is distributed as $\mathbb{P}^{u_0^{n_1}}$. Again, starting from such potential there is probability at least δ that (4) holds for all j , with $(u_n^0)_n$ replaced by $(u_n^{n_1})_n$. This can be repeated indefinitely, and therefore there will almost surely be a time n_* such that for condition (4) holds for all j , with $(u_n^0)_n$ replaced by $(u_n^{n_*})_n$.

By definition of ℓ and L , we have $L_j \sim \frac{48j^{5/4}}{5}$ and $\ell_j/L_j \sim \frac{5}{4j}$. Now, by construction of N , $L_j \leq N_j \leq L_j + m(u_0^{n*})$, and therefore $N_{j+1} \sim N_j \sim L_j$. Finally, assuming that (4) holds for all j , $X_j \sim_\epsilon \ell_j/L_j$.

But $Z_{j+1} = \sum_{i=1}^j X_i$, and putting these all together gives

$$Z_{j+1} \sim_\epsilon \frac{5}{4} \log j$$

Finally, the position S_n is given by $S_n = S_{n_*} + \sigma Z_j$ at times n satisfying $n = n_* + L_j$, and therefore

$$S_n \sim_\epsilon \sigma \log n \quad \text{a.s.}$$

Since ϵ was arbitrary,

$$\frac{S_n}{\sigma \log n} \rightarrow 1 \quad \text{a.s.}$$

and using Lemma 1 this finishes the proof of Theorem 1. \square

3 Block argument

In this section we prove Proposition 1. Let $0 < \epsilon < \frac{1}{2}$.

At step $j = 0$, we find the first customer S_1 , take $\sigma = \text{sgn}(S_1)$ and let $Z_1 = \sigma S_1 = |S_1|$. The next steps $j = 1, 2, 3, \dots$ are described assuming that $\sigma = +1$.

We are going to define the event A_j that step j is successful. For each j , the occurrence of A_j implies (4), and we will show that there exists a sequence p_0, p_1, p_2, \dots , depending only on ϵ , such that

$$\mathbb{P}^u \left(A_j \mid A_{j-1}, A_{j-2}, \dots, A_0 \right) \geq p_j \quad (5)$$

and

$$\prod_{j=0}^{\infty} p_j > 0. \quad (6)$$

For the latter we show that p_j increases fast enough so that $1 - p_j$ is summable, and that $p_j > 0$ for all j . Let us drop the superscript 0 in the potentials u_n^0 .

We start with $j = 0$. Take $L_1 = Q_0 = \ell_0 = 1$, $Z_1 = X_0 = \sigma S_1 = |S_1|$, and $N_1 = L_1 - u(Z_1)$. We say that Step 0 is *successful* if

$$X_0 \geq X_0^- := \frac{3}{N_1}, \quad (7)$$

otherwise we declare Step 0 to have *failed* and stop. Without loss of generality assume that $\sigma = +1$.

Suppose that Steps $0, 1, 2, \dots, j-1$ have been successful and start from u_{L_j} . Step j may be successful in two situations. First, if each of the next ℓ_j customers $S_{L_j+1}, S_{L_j+2}, \dots, S_{L_j+\ell_j}$ satisfy $S_n > S_{n-1}$, in which case we take $Q_j = \ell_j$. Second, if there is one $\tilde{n} \in \{L_j + 1, \dots, L_j + \ell_j\}$ such that $S_{\tilde{n}} < S_{\tilde{n}-1}$, and $S_n > S_{n-1}$ for all $n \in \{L_j + 1, \dots, L_j + \ell_j, L_j + \ell_j + 1\}$ except \tilde{n} , in which case we take $Q_j = \ell_j + 1$. If none of these two happen, we declare Step j to have *failed* and stop. Otherwise, in either of the above two cases we say that Step j is *successful* if (4) is satisfied.

Notice that, for $j \geq 1$, if Step $j-1$ is successful we have

$$\begin{cases} \mathcal{M}(u_{L_j}) = L_j, \\ u_{L_j}(x) = u_0(x) \leq L_j - N_j & \text{for } x > Z_j, \\ u_{L_j}(x) \geq L_j - Q_{j-1} & \text{for } Z_j - X_{j-1}^- < x < Z_j. \end{cases} \quad (8)$$

Having described the grouping steps, it remains to show (5) and (6).

Recall from the previous section that, once u_n is fixed, the position of the next customer S_{n+1} is determined by a pair E_{n+1}, U_{n+1} of exponentially- and uniformly-distributed random variables, or alternatively by the Poisson point process ν restricted to the region $\{(x, t) : u_n(x) < t \leq \mathcal{M}(u_n)\}$.

We start with $j = 0$. In this step we pay a *finite price* p_0 to produce a potential which exhibits a *plateau* with convenient shape, namely a potential satisfying (7). Recall that E_1 and U_1 are the exponential and uniform random variables used in order to produce u_1^0 from u_0^0 . Consider the event that E_1 and U_1 satisfy the following requirements. First, that $E_1 > 6$ and second, that U_1 lies on the largest interval among $[0, \frac{a}{a+b}]$ and $[\frac{a}{a+b}, 1]$; see (1). In the worst case this interval has length $\frac{1}{2}$, whence the probability that both conditions are satisfied is at least $p_0 = \frac{1}{2}e^{-6} > 0$. The requirement for U_1 implies that $u(S_1) \leq u(-S_1)$. Hence, by monotonicity of u_0^0 , the occurrence of the above event implies that

$$6 < \int_{-S_1}^{+S_1} -u(x)dx \leq \int_{-S_1}^{+S_1} \max_{[-S_1, +S_1]} (-u)dx = -2X_0u(S_1) \leq 2X_0N_1.$$

The above inequality implies A_0 , and therefore $\mathbb{P}^u(A_0) \geq p_0 > 0$.

Fix some $j = 1, 2, 3, \dots$. We will describe a number of events, which we denote by B_1, B_2, B_3 , omitting the dependency on j , such that $B_1 \cap B_2 \cap B_3$ implies A_j . The

conditional probability of $B_1 \cap B_2 \cap B_3$ given u_{L_j} can be bounded from below by some number p_j that does not depend on the potential u_{L_j} as long as it satisfies (8). This in turn implies (5).

We stress that, even though the knowledge about these events inconveniently provides more information about ν than needed in determining $u_{L_{j+1}}$, we only study them with the purpose of estimating the probability of A_j . The occurrence of the latter is entirely determined by $u_{L_j}, u_{L_j+1}, u_{L_j+2}, \dots, u_{L_{j+1}}$.

We consider the evolution given by the point process ν itself rather than the construction specified in (1). We write $\nu_i = \nu \cap R_i$, where

$$\begin{aligned} R_1 &= \{(x, t) : x > Z_j, u(x) < t \leq L_j\}, \\ R_2 &= \{(x, t) : Z_j < x < Z_j + X_j^+, L_j < t \leq L_j + Q_j^+\} \cup \\ &\quad \cup \{(x, t) : Z_j - X_{j-1}^- < x < Z_j, u_{L_j}(x) < t \leq L_j + Q_j^+\}. \end{aligned}$$

The first event considered is

$$B_1 := \left[|\nu_2| \leq 1 \right].$$

Notice that, conditioned on u_{L_j} , the number of points $|\nu_2|$ is distributed as a Poisson random variable with mean given by the area $|R_2|$. Now, on the event that u_{L_j} satisfies (8),

$$|R_2| \leq Q_j^+ X_j^+ + Q_{j-1}^+ X_{j-1}^- + Q_j^+ X_{j-1}^- \leq 3Q_j^+ X_j^+ \leq C \frac{(\ell_j + 1)^2}{N_j} \leq C \frac{1}{j^{3/4}}$$

since

$$\ell_j \leq Cj^{1/4} \quad \text{and} \quad N_j \geq L_j \geq Q_0 + \dots + Q_{j-1} \geq Cj^{5/4},$$

and therefore

$$\mathbb{P}(B_1 | u_{L_j}) \geq 1 - C|R_2|^2 \geq 1 - C \frac{1}{j^{3/2}}.$$

We also need the estimate to be positive for all j , which follows from

$$\mathbb{P}(B_1 | u_{L_j}) \geq \mathbb{P}(\nu_2 = \emptyset | u_{L_j}) = e^{-|R_2|} \geq e^{-c} > 0.$$

We now consider the events B_2 and B_3 , which depend on ν_1 . Define

$$A(x) = \int_{Z_j}^x [L_j - u(z)] dz, \quad x \geq Z_j,$$

and write $\nu_1 = \{(x_1, t_1), (x_2, t_2), (x_3, t_3), \dots\}$ with $x_0 = Z_j < x_1 < x_2 < x_3 < \dots$. By definition of ν_1 , we have that $(A(x_n) - A(x_{n-1}))_{n=1,2,3,\dots}$ are i.i.d. exponential

random variables with mean 1, independent of u_{L_j} . The events B_2 and B_3 are defined in terms of $A(x_n)$, $n = 1, 2, 3, \dots$, whence the estimates on their probabilities are always uniform on u_{L_j} .

Consider the event

$$B_2 := \left[(1 - \epsilon)(\ell_j - 1) < A(x_{\ell_j-1}) < A(x_{\ell_j}) < (1 + \epsilon)\ell_j \right]. \quad (9)$$

By Cramér's large deviation principle

$$\mathbb{P}(B_2) \geq 1 - e^{-C_\epsilon \ell_j}. \quad (10)$$

Let

$$D_j = \frac{\ell_j}{12} \quad (11)$$

and consider the event

$$B_3 := \left[A(x_n) - A(x_{n-1}) \leq D_j \text{ for } n = 1, 2, \dots, \ell_j \right]. \quad (12)$$

By a simple union bound we have

$$\mathbb{P}(B_3) \geq 1 - \ell_j e^{-D_j} \geq 1 - ce^{-C_\epsilon \ell_j}. \quad (13)$$

Using (10) and (13) we get

$$\mathbb{P}(B_2 \cap B_3) \geq 1 - ce^{-C_\epsilon \ell_j}.$$

Now, since $D_j \geq 1$, we have

$$\mathbb{P}(B_2 \cap B_3) \geq \mathbb{P}(1 - \epsilon < A(x_n) - A(x_{n-1}) < 1 \text{ for } n = 1, 2, \dots, \ell_j) > e^{-C_\epsilon \ell_j} > 0$$

and thus adjusting C_ϵ we get

$$\mathbb{P}(B_2 \cap B_3) \geq 1 - e^{-C_\epsilon \ell_j}.$$

Since ν_1 is conditionally independent of $\nu_2 \cup \nu_3$ given u_{L_j} , we have that

$$\mathbb{P}(B_1 \cap B_2 \cap B_3 | u_{L_j}) \geq p_j$$

for

$$p_j = (1 - e^{-C_\epsilon \ell_j}) (e^{-c} \vee (1 - Cj^{-3/2})).$$

Notice that the sequence $(p_j)_{j=0,1,2,\dots}$ satisfies (6), thus it only remains to show that $B_1 \cap B_2 \cap B_3$ implies A_j .

Suppose B_1 , B_2 , and B_3 happen. By (8) and monotonicity of u we have

$$N_j[x_n - x_{n-1}] \leq A(x_n) - A(x_{n-1}) \leq [L_j - u(x_n)] [x_n - x_{n-1}],$$

whence by (9)

$$x_{\ell_j-1} - Z_j \leq x_{\ell_j} - Z_j \leq (1 + \epsilon) \frac{\ell_j}{N_j} = X_j^+, \quad (14)$$

and by (12)

$$x_n - x_{n-1} \leq \frac{D_j}{N_j} \leq \frac{X_{j-1}^-}{3}. \quad (15)$$

Moreover, for $n = 1, 2, \dots, \ell_j - 1$,

$$A(x_n) - A(x_{n-1}) \leq [L_j - u(x_{\ell_j-1})] [x_n - x_{n-1}]$$

and, by (9),

$$x_{\ell_j} - Z_j \geq x_{\ell_j-1} - Z_j \geq (1 - \epsilon) \frac{\ell_j - 1}{L_j - u(x_{\ell_j-1})} \geq (1 - \epsilon) \frac{\ell_j - 1}{N_{j+1}} = X_j^-$$

as long as $Z_{j+1} = S_{L_j+Q_j} \geq x_{\ell_j-1}$.

Therefore, to prove (4) it suffices to show that

$$\begin{cases} x_{\ell_j-1} \leq S_{L_j+Q_j} \leq x_{\ell_j} \\ x_0 - X_{j-1}^- \leq S_{L_j+n} < S_{L_j+Q_j}, \quad n = 1, 2, \dots, Q_j - 1. \end{cases} \quad (16)$$

The remainder of the proof is dedicated to proving (16) assuming (14), (15), and that B_1 occurs.

We first recall that the points in $(x, t) \in \nu$ that correspond to customers $(S_{L_j+1}, S_{L_j+2}, \dots, S_{L_j+Q_j})$ are such that $u_{L_j}(x) < t \leq L_j + Q_j^+$. When these points are neither in R_1 nor in R_2 , they must be in R_3 given by $t \in (u_{L_j}(x), L_j + Q_j^+]$ and

$$x < Z_j - X_{j-1}^- \quad \text{or} \quad x > Z_j + X_j^+. \quad (17)$$

The points in R_1 are given by $(x_n, t_n)_{n=1,2,\dots}$, and R_2 is either empty or contains one point, denoted by (x', t') .

Let n' be the maximal index between 0 and ℓ_j such that

$$(S_{L_j}, S_{L_j+1}, S_{L_j+2}, \dots, S_{L_j+n'-1}, S_{L_j+n'}) = (x_0, x_1, x_2, \dots, x_{n'-1}, x_{n'}).$$

If $n' = \ell_j$, we have $Q_j = \ell_j$, thus (16) is satisfied. So suppose $n' \leq \ell_j - 1$. We claim that

$$S_{L_j+n'+1} = x'$$

with x' satisfying

$$x_{n'} - \frac{D_j}{N_j} \leq x' < x_{n'+1},$$

and moreover

$$S_{L_j+n+1} = x_n \text{ for } n = n' + 1, n' + 2, \dots, \ell_j, \quad (18)$$

i.e., the points in R_3 cannot participate the construction of $S_{L_j+Q_j}$.

In the case $x' < x_{n'}$, we will have $Q_j = \ell_j + 1$ and $S_{L_j+Q_j} = x_{\ell_j}$. Otherwise, $x_{n'} < x' < x_{n'+1}$, we will have $Q_j = \ell_j$, and in this case $S_{L_j+Q_j} = x_{\ell_j-1}$ if $n' \leq \ell_j - 2$ or $S_{L_j+Q_j} = x' \in (x_{\ell_j-1}, x_{\ell_j})$ if $n' = \ell_j - 1$. Therefore (16) is always satisfied.

It thus remains to prove the above claim. By definition of n' , the point $(x', t') \in \nu$ corresponding to $S_{L_j+n'+1}$ cannot be in R_1 . But it cannot be in R_3 either. Indeed, since $S_{L_j+n'} = x_{n'}$ and

$$x_{n'} < x_{n'+1} \leq x_{n'} + \frac{D_j}{N_j},$$

we must have

$$x_0 - \frac{D_j}{N_j} \leq x_{n'} - \frac{D_j}{N_j} \leq x' < x_{n'+1},$$

thus x' cannot satisfy (17). Therefore, (x', t') is the only point in ν_2 .

We finally show (18). Start with $n = n' + 1$. Write $\tilde{x} = S_{L_j+n'+2}$, corresponding to a point $(\tilde{x}, \tilde{t}) \in \nu$. This point cannot be in R_2 , since (x', t') was the only such point. As before,

$$|x' - x_{n'+1}| \leq |x' - x_{n'}| + |x_{n'} - x_{n'+1}| \leq 2\frac{D_j}{N_j},$$

thus we must have

$$\tilde{x} < x_{n'+1} \leq x_{\ell_j} \leq Z_j + X_j^+,$$

and

$$|\tilde{x} - x'| < 2\frac{D_j}{N_j},$$

whence

$$\tilde{x} > x' - 2\frac{D_j}{N_j} \geq x_0 - 3\frac{D_j}{N_j}$$

and again \tilde{x} cannot satisfy (17) either. Therefore, $(\tilde{x}, \tilde{t}) \in \nu_1$ which implies $\tilde{x} = x_{n'+1}$. For $n = n' + 2, \dots, \ell_j$ the argument is the same.

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